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Alternative structures and bi-Hamiltonian systems

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Abstract

In the study of bi-Hamiltonian systems (both classical and quantum) one starts with a given dynamics and looks for all alternative Hamiltonian descriptions it admits. In this paper, we start with two compatible Hermitian structures (the quantum analogue of two compatible classical Poisson brackets) and look for all the dynamical systems which turn out to be bi-Hamiltonian with respect to them.

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1. Introduction

It is by now well known that the general structures of classical and quantum systems are not essentially different. When considered as abstract dynamical systems on infinite- or finite-dimensional vector spaces of states, both are 'Hamiltonian vector fields', and when considered in the Schrödinger picture, both are 'inner derivations' on the Lie algebra of observables with respect to the Poisson brackets and the commutator brackets respectively [1]. Moreover, in some appropriate limit, quantum mechanics should reproduce classical mechanics. From this point of view, it is a natural question to ask which alternative quantum descriptions of a given quantum system would reproduce the alternative classical descriptions known as bi-Hamiltonian descriptions of integrable systems [2]. This question has been addressed recently in several collaborations involving some of us in different combinations [3].

When we consider composite systems and interactions among them it is interesting to analyse to what extent these alternative quantum descriptions survive. This paper is a preliminary step in this direction.

The specific problem we would like to address can then be stated as follows:

'How many different quantum systems may, at the same time, admit to bi-Hamiltonian descriptions with respect to the same alternative structures?'

We will show that for generic, compatible, alternative structures the different admissible quantum systems are pairwise commuting; moreover, in finite dimensions, they generate a maximal torus.

To tackle the stated problem [4], we construct a quantum system out of a given Hermitian structure and then we consider compatible systems out of two Hermitian structures. The 'quantum systems' associated with a given Hermitian structure correspond to the infinitesimal generator of the 'phase group'. The compatibility condition amounts to the requirement of compatibility (commutativity) of the 'phase groups' with respect to both Hermitian structures.

All of our considerations will be carried over on finite-dimensional vector spaces, and only in the final section shall we consider the extension of our results to infinite-dimensional Hilbert spaces.

2. Hermitian structures on R^{2n}

In view of our interest in quantum systems, we will focus our attention here on linear systems. Also, the tensorial structures that are descried below will be assumed to be represented by constant matrices. We will consider, to start with, three relevant tensor structures that can be introduced in R^{2n}, namely a *metric tensor* of the form

$$g = g_{jk}\,\mathrm{d}x^j \otimes \mathrm{d}x^k \tag{1}$$

a *symplectic structure*

$$\omega = \omega_{jk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k \tag{2}$$

and a *complex structure* J, i.e. a (1, 1)-type tensor satisfying

$$J^2 = -\mathbb{I}. \tag{3}$$

We will be interested in the case in which the above structures are *admissible*. By this we mean the following:

(i) Suppose we are given g and J. We will say that they are admissible, or that g is 'Hermitian' iff

$$g(Jx, Jy) = g(x, y) \qquad \forall x, y. \tag{4}$$

We will always assume this to be the case.

Now, by virtue of $J^2 = -\mathbb{I}$, $g(x, Jy) = -g(J(Jx), Jy) = -g(Jx, y)$. Hence

$$g(Jx, y) + g(x, Jy) = 0. \tag{5}$$

Note that the previous two equations imply that J generates *finite* as well as *infinitesimal* rotations at the same time. Moreover, equation (5) implies that the (0, 2)-type tensor

$$\omega =: g(J\cdot, \cdot) \tag{6}$$

is a symplectic form on R^{2n} and that

$$J = g^{-1} \circ \omega. \tag{7}$$

By proceeding as before, one can prove that

$$\omega(Jx, Jy) = \omega(x, y) \tag{8}$$

and

$$\omega(Jx, y) + \omega(x, Jy) = 0. \tag{9}$$

J will then generate both finite and infinitesimal symplectic transformations.

(ii) Alternatively, one could start from g and ω and say that they are admissible iff $J = g^{-1} \circ \omega$ satisfies $J^2 = -\mathbb{I}$. Written explicitly in components, this condition reads

$$g^{jk}\omega_{kl}\, g^{lm}\omega_{mn} = -\delta^j_n. \tag{10}$$

Remark.

(i) Note that if condition (4) does not hold, we can always build a Hermitian structure out of a given g by substituting it with the symmetrized metric tensor

$$g_s(\cdot,\cdot) =: \tfrac{1}{2}\{g(J\cdot, J\cdot) + g(\cdot,\cdot)\} \tag{11}$$

which will be positive and nondegenerate if g is positive and nondegenerate.

Quite similarly [5], if condition (10) does not hold, then Riesz's theorem tells us that there exists a nonsingular linear operator A such that

$$\omega(x, y) = g(Ax, y) \tag{12}$$

and the antisymmetry of ω implies

$$g(Ax, y) = -g(x, Ay) \tag{13}$$

i.e. A is skew-Hermitian $A^\dagger = -A$, and $-A^2 > 0$. Let P then be a (symmetric) non-negative square root of A. P will be injective, and so P^{-1} will be well defined[5]. We then define $J =: AP^{-1}$ and

$$g_\omega(\cdot,\cdot) =: g(P(\cdot),\cdot). \tag{14}$$

Therefore

$$\omega(x, y) = g(Ax, y) = g_\omega(Jx, y) \tag{15}$$

and $J^\dagger = -J$, $J^2 = -\mathbb{I}$. The triple (g_ω, J, ω) will then be an admissible triple, equation (5) will hold true for g_ω and, moreover,

$$g_\omega(Jx, Jy) = g(Ax, Jy) = -g(AJx, y) = g_\omega(x, y) \tag{16}$$

and equation (4) will be satisfied as well.

(ii) The adjoint $A^\dagger$ of any linear operator A w.r.t. a metric tensor g is defined by the standard relation

$$g(A^\dagger x, y) =: g(x, Ay) \tag{17}$$

and we can read equation (5) as saying that the complex structure J is skew-adjoint w.r.t. the metric tensor g.

Although it may seem elementary, it is worth stressing here that despite the fact that we are working in a real vector space, the adjoint of A does *not* coincide with the transpose A^T for a general g. Indeed, spelling out equation (17) explicitly in terms of matrices leads to

$$A^\dagger = g^{-1}A^T g \tag{18}$$

and therefore, even for real matrices, $A^\dagger = A^T$ only if the metric is standard Euclidean, and in general, symmetric matrices need not be self-adjoint.

A linear structure on R^{2n} is associated with a given dilation (or Liouville) vector field Δ. Given then a linear structure on $\mathbb{R}^{2n}$, we can associate with every matrix $\mathbb{A} \equiv \|A^i{}_j\| \in \mathfrak{gl}(2n, \mathbb{R})$ both a (1, 1)-type tensor field

$$T_{\mathbb{A}} = A^i_j\, \mathrm{d}x^j \otimes \frac{\partial}{\partial x^i} \tag{19}$$

[5] In the infinite-dimensional case, it can be proved [5] that A is bounded and injective, and that P is also injective and densely defined, so that P^{-1} is well defined in the infinite-dimensional case as well.

and a linear vector field

$$X_{\mathbb{A}} = A^i_j x^j \frac{\partial}{\partial x^i}. \tag{20}$$

The two are connected by

$$T_{\mathbb{A}}(\Delta) = X_{\mathbb{A}} \tag{21}$$

and are both homogeneous of degree zero, i.e.

$$L_\Delta X_{\mathbb{A}} = L_\Delta T_{\mathbb{A}} = 0. \tag{22}$$

The correspondence $\mathbb{A} \to T_{\mathbb{A}}$ is (full) associative algebra and Lie algebra isomorphism. The correspondence $\mathbb{A} \to X_{\mathbb{A}}$ is instead only a Lie algebra (anti)isomorphism, i.e.

$$T_{\mathbb{A}} \circ T_{\mathbb{B}} = T_{\mathbb{A}\mathbb{B}} \tag{23}$$

while

$$[X_{\mathbb{A}}, X_{\mathbb{B}}] = -X_{[\mathbb{A},\mathbb{B}]}. \tag{24}$$

Moreover, for any $\mathbb{A}, \mathbb{B} \in \mathfrak{gl}(2n, \mathbb{R})$

$$L_{X_{\mathbb{A}}} T_{\mathbb{B}} = -T_{[\mathbb{A},\mathbb{B}]}. \tag{25}$$

This implies that all statements that can be proved at the level of vector fields and/or at that of $(1, 1)$ tensors can be rephrased into equivalent statements in terms of the corresponding representative matrices, and vice versa. That is why we will work mostly directly with the representative matrices in what follows.

Out of the Liouville field and the metric tensor, we can construct the quadratic function

$$\mathsf{g} = \tfrac{1}{2} g(\Delta, \Delta) \tag{26}$$

and the associated Hamiltonian vector field Γ via

$$i_\Gamma \omega = -\mathrm{d}\mathsf{g}. \tag{27}$$

In a given coordinate system $(x^1, \cdot, x^{2n})$, $\Delta = x^i \partial/\partial x^i, \Gamma = \Gamma^k \partial/\partial x^k$ and explicitly, $\omega_{jk}\Gamma^k = -g_{jk}x^k$, implying $g^{lj}\omega_{jk}\Gamma^k = -x^l$, i.e. $\Gamma^k = J^k_l x^l$ or

$$J(\Gamma) = -\Delta \quad \Leftrightarrow \quad \Gamma = J(\Delta). \tag{28}$$

Γ will preserve both g and ω, and hence J (more generally, it will preserve any third structure if it preserves the other two):

$$L_\Gamma \omega = L_\Gamma g = L_\Gamma J = 0. \tag{29}$$

Given a metric tensor and an admissible symplectic form, a *Hermitian structure* on $\mathbb{R}^{2n}$ is a map $h : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^2$ defined as

$$h : (x, y) \mapsto (g(x, y), \omega(x, y)).$$

Equivalently (and having in mind quantum systems) one can exploit the fact that R^{2n} can be given a complex vector space structure by defining, for $z = \alpha + \mathrm{i}\beta \in \mathbb{C}$ and $x \in \mathbb{R}^{2n}$,

$$z \cdot x = (\alpha + \mathrm{i}\beta)x =: \alpha x + J\beta x. \tag{30}$$

Then h will become a Hermitian scalar product, linear in the second factor, on a complex vector space, and we can now write

$$h(x, y) = g(x, y) + \mathrm{i}g(Jx, y) \tag{31}$$

and of course a statement equivalent to the previous ones will be

$$L_\Gamma h = 0. \tag{32}$$

The vector field Γ will therefore be a generator of the unitary group on C^n, and will be an instance of what we will call a *quantum system*. More generally, a quantum system will be any linear vector field

$$\Gamma_{\mathbb{A}} = A^j_k x^k \frac{\partial}{\partial x^j} \tag{33}$$

associated with a matrix $\mathbb{A} = \|A^j{}_k\|$ that preserves both g and ω or, equivalently, h:

$$L_{\Gamma_{\mathbb{A}}} h = 0. \tag{34}$$

This defining requirement on $\Gamma_{\mathbb{A}}$ implies that the matrix $\mathbb{A}$ in the description of $\Gamma_{\mathbb{A}}$ be at the same time an infinitesimal generator of a realization of the symplectic group $Sp(n)$ and of a realization of the rotation group $SO(2n)$. The intersection of these two Lie algebras yields a realization of the Lie algebra of the unitary group.

3. Bi-Hamiltonian descriptions

Consider now two different Hermitian structures on R^{2n}, $h_1 = g_1 + \mathrm{i}\omega_1$ and $h_2 = g_2 + \mathrm{i}\omega_2$, with the associated quadratic functions $\mathsf{g}_1 = g_1(\Delta, \Delta)$, $\mathsf{g}_2 = g_2(\Delta, \Delta)$ and Hamiltonian vector fields Γ_1 and Γ_2. Then

Definition. *h_1 and h_2 will be said to be 'compatible' iff*

$$L_{\Gamma_1} h_2 = L_{\Gamma_2} h_1 = 0. \tag{35}$$

This will of course imply

$$L_{\Gamma_1} \omega_2 = L_{\Gamma_1} g_2 = 0 \tag{36}$$

separately, and similar equations with the indices interchanged.

Remark. Note that if $\omega = \frac{1}{2}\omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ is a constant symplectic structure and $X = A^i_j x^j \partial\big/\partial x^i$ is a linear vector field, then the condition $L_X\omega = 0$ can be written in terms of the representative matrices as the requirement that the matrix ωA is symmetric, i.e.

$$\omega A = (\omega A)^T \quad \Leftrightarrow \quad \omega A + A^T\omega = 0 \tag{37}$$

while the condition $L_X g = 0$ implies that the matrix gA is skew-symmetric, i.e.

$$gA + (gA)^T = gA + A^T g = 0. \tag{38}$$

Note now that from $i_{\Gamma_2}\omega_2 = -\mathrm{d}\mathsf{g}_2$ and $L_{\Gamma_1} g_2 = 0$, we obtain

$$0 = L_{\Gamma_1}\big(i_{\Gamma_2}\omega_2\big) =: L_{\Gamma_1}\omega_2(\Gamma_2, \cdot) = \big(L_{\Gamma_1}\omega_2\big)(\Gamma_2, \cdot) + \omega_2([\Gamma_1, \Gamma_2], \cdot)$$

and as $L_{\Gamma_1}\omega_2 = 0$

$$i_{[\Gamma_1,\Gamma_2]}\omega_2 = 0 \tag{39}$$

and similarly for ω_1. As neither ω_1 nor ω_2 is degenerate, this implies that Γ_1 and Γ_2 commute, i.e.

$$[\Gamma_1, \Gamma_2] = 0. \tag{40}$$

Moreover, remembering that given a symplectic structure ω, the Poisson bracket of any two functions f and g is defined as $\{f, g\} =: \omega(X_g, X_f)$, where X_f and X_g are the Hamiltonian vector fields associated with f and g respectively, we find $0 = L_{\Gamma_1}\mathsf{g}_2 = \mathrm{d}\mathsf{g}_2(\Gamma_1) = -\omega_2(\Gamma_2, \Gamma_1)$. Hence we find

$$\{\mathsf{g}_1, \mathsf{g}_2\}_2 = 0 \tag{41}$$

where $\{\cdot, \cdot\}_2$ is the Poisson bracket associated with ω_2 and similarly for ω_1.

Remark. The four real conditions, $\{\mathrm{g}_1, \mathrm{g}_2\}_{1,2} = 0$ and $L_{\Gamma_1}\omega_2 = L_{\Gamma_2}\omega_1 = 0$, are actually equivalent to those stated in complex form in equation (35).

Remembering what has already been stated about the fact that statements concerning linear vector fields translate into equivalent statements for the (1, 1)-type tensor fields having the same representative matrices, and recalling that the defining matrices of Γ_1 and Γ_2 are precisely those of the corresponding complex structures, we see at once that

$$[\Gamma_1, \Gamma_2] = 0 \quad \Leftrightarrow \quad [J_1, J_2] = 0 \tag{42}$$

i.e. the two complex structures will commute as well.

In general, given any two (0, 2) (or (2, 0)) tensor fields h and g one (at least) of which, say h, is invertible, the composite tensor $h^{-1} \circ g$ will be a (1, 1)-type tensor. Then, out of the two compatible structures, we can build up the two (1, 1)-type tensor fields:

$$G = g_1^{-1} \circ g_2 \tag{43}$$

and

$$T = \omega_1^{-1} \circ \omega_2. \tag{44}$$

Actually, one can prove at once that the two are related, and indeed direct calculation proves that

$$G = J_1 \circ T \circ J_2^{-1} \equiv -J_1 \circ T \circ J_2 \quad \Leftrightarrow \quad T = -J_1 \circ G \circ J_2. \tag{45}$$

It turns out that T (and hence G) commutes with both complex structures, i.e.

$$[G, J_a] = [T, J_a] = 0 \qquad a = 1, 2. \tag{46}$$

This follows from the fact that both G and T are Γ-invariant, i.e.

$$L_{\Gamma_{1,2}} G = L_{\Gamma_{1,2}} T = 0 \tag{47}$$

and from equation (25).

It follows also from equations (25) and (26) that G and T commute, i.e.

$$[G, T] = 0. \tag{48}$$

Moreover, G enjoys the property that

$$g_a(Gx, y) = g_a(x, Gy) \qquad a = 1, 2. \tag{49}$$

Indeed, one can prove by direct calculation that

$$g_1(Gx, y) = g_1(x, Gy) = g_2(x, y) \tag{50}$$

while

$$g_2(Gx, y) = g_2(x, Gy) = g_1^{-1}(g_2(x, \cdot), g_2(y, \cdot)) \tag{51}$$

and this completes the proof. In equation (49) it can be seen that G is 'self-adjoint' w.r.t. both metrics.

Note that the derivation of this result does not require the compatibility condition to hold. If the latter is assumed, however, one can also prove that T is self-adjoint w.r.t. both metrics, and that both J_1 and J_2 are instead skew-adjoint w.r.t. both structures, i.e.

$$g_a(Tx, y) = g_a(x, Ty) \qquad a = 1, 2 \tag{52}$$

and

$$g_1(x, J_2y) + g_1(J_2x, y) = 0 \qquad \forall x, y \tag{53}$$

with a similar equation with the indices interchanged.

Indeed, from, e.g., $L_{\Gamma_1}\omega_2 = 0$ we obtain, in terms of the representative matrices and using equation (37) and $J_1 = g_1^{-1}\omega_1$,

$$\omega_2 g_1^{-1}\omega_1 = \omega_1 g_1^{-1}\omega_2 \quad \Leftrightarrow \quad \omega_2\omega_1^{-1}g_1 = g_1\omega_1^{-1}\omega_2. \tag{54}$$

Remembering the definition of T, this is equivalent to $g_1 T = (g_1 T)^T$, which leads to

$$T = g_1^{-1}T^T g_1 = (T^\dagger)_1 \tag{55}$$

where $(T^\dagger)_1$ is the adjoint of T w.r.t. g_1. Interchanging indices, one can prove that $(T^\dagger)_2 = T$ as well.

Concerning J (that have already been proved to be skew-adjoint w.r.t. the respective metric tensors), consider, e.g.

$$(J_1^\dagger)_2 =: g_2^{-1}J_1^T g_2 = -g_2^{-1}g_1 J_1^T g_1^{-1}g_2 = -G^{-1}J_1 G = -J_1$$

as G and J commute. A similar result holds of course for J_2.

Summarizing what has been proved up to now, we have found that G, T, J_1 and J_2 are a set of mutually commuting linear operators. G and T are self-adjoint while J_1 and J_2 are skew-adjoint, w.r.t. both metric tensors.

If we now diagonalize G, the $2n$-dimensional vector space $V = \mathbb{R}^{2n}$ will split into a direct sum of eigenspaces, $V = \oplus_k V_{\lambda_k}$, where λ_k $(k = 1, \ldots, r \leqslant 2n)$ are the distinct eigenvalues of G. According to what has just been proved, the sum will be an orthogonal sum w.r.t. both metrics, and, in V_{λ_k}, $G = \lambda_k\mathbb{I}_k$ with $\mathbb{I}_k$ the identity matrix in V_{λ_k}. Assuming compatibility, T will commute with G and will be self-adjoint. Therefore, we will get a further orthogonal decomposition of each V_{λ_k} of the form

$$V_{\lambda_k} = \bigoplus_r W_{\lambda_k,\mu_{k,r}} \tag{56}$$

where $\mu_{k,r}$ are the (distinct) eigenvalues of T in V_{λ_k}. The complex structures commute in turn with both G and T. Therefore they will leave each one of $W_{\lambda_k,\mu_{k,r}}$ invariant.

Now we can reconstruct, using g and J, the two symplectic structures. They will be block-diagonal in the decomposition (47) of V, and on each one of $W_{\lambda_k,\mu_{k,r}}$ they will be of the form

$$g_1 = \lambda_k g_2 \qquad \omega_1 = \mu_{k,r}\omega_2. \tag{57}$$

Therefore, in the same subspaces $J_1 = g_1^{-1}\omega_1 = \frac{\mu_{k,r}}{\lambda_k}J_2$. It follows from $J_1^2 = J_2^2 = -1$ that $\left(\frac{\mu_{k,r}}{\lambda_k}\right)^2 = 1$, whence $\mu_{k,r} = \pm\lambda_k$ (and $\lambda_k > 0$). The index r can then assume only two values, corresponding to $\pm\lambda_k$ and at most V_{λ_k} will have the decomposition of V_{λ_k} into the orthogonal sum, $V_{\lambda_k} = W_{\lambda_k,\lambda_k} \oplus W_{\lambda_k,-\lambda_k}$. All in all, what we have proved is the following:

Lemma. *If the two Hermitian structures $h_1 = (g_1, \omega_1)$ and $h_2 = (g_2, \omega_2)$ are compatible*[6]*, then the vector space $V \approx \mathbb{R}^{2n}$ will decompose into the (double) orthogonal sum:*

$$\bigoplus_{k=1,\ldots,r;\,\alpha=\pm} W_{\lambda_k,\alpha\lambda_k} \tag{58}$$

where the index $k = 1, \ldots, r \leqslant 2n$ labels the eigenspaces of the (1, 1)-type tensor $G = g_1^{-1}\circ g_2$ corresponding to its distinct eigenvalues $\lambda_k > 0$, while $T = \omega_1^{-1} \circ \omega_2$ will be diagonal

[6] Coming, of course, from admissible triples (g_1, ω_1, J_1) and (g_2, ω_2, J_2).

(with eigenvalues $\pm\lambda_k$) on $W_{\lambda_k,\pm\lambda_k}$, on each one of which

$$g_1 = \lambda_k g_2 \qquad \omega_1 = \pm\lambda_k\omega_2 \qquad J_1 = \pm J_2. \tag{59}$$

As neither symplectic form is degenerate, the dimension of each one of $W_{\lambda_k,\pm\lambda_k}$ will be necessarily even.

Now we can further qualify and strengthen the compatibility condition by stating the following:

Definition. *Two (compatible) Hermitian structures will be said to be 'in a generic position' iff the eigenvalues of G and T have minimum (i.e. double) degeneracy.*

In general, two appropriate geometrical objects such as two (0, 2) or (2, 0)-type tensor fields are said to be in a generic position if they can be 'composed' to yield a 1–1 tensor whose eigenvalues have minimum degeneracy. For instance, g_1 and g_2 are in a generic position if the eigenvalues of $G = g_1^{-1} \circ g_2$ have minimum degeneracy, which possibly depends on further conditions: when the compatibility is required, this degeneracy is double. The results that we have just proved will imply that each one of W_{λ_k,λ_k}, $W_{\lambda_k,-\lambda_k}$ will have the minimum possible dimension, that is 2.

Denoting these two-dimensional subspaces then by E_k ($k = 1, \ldots, n$, now), all that has been said up to now can be summarized in the following:

Proposition. *If h_1 and h_2 are compatible and in a generic position, then $\mathbb{R}^{2n}$ splits into a sum of n mutually 'bi-orthogonal' (i.e. orthogonal with respect to both metrics g_1 and g_2) two-dimensional vector subspaces: $\mathbb{R}^{2n} = E_1 \oplus E_2 \oplus \cdots \oplus E_n$. All the structures g_a, J_a, ω_a decompose accordingly into a direct sum of structures on these two-dimensional subspaces, and on each one of E_k they can be written as*

$$\begin{array}{llll} g_1|_{E_k} = \lambda_k(e_1^* \otimes e_1^* + e_2^* \otimes e_2^*) & \lambda_k > 0 & g_2|_{E_k} = \varrho_k g_1|_{E_k} & \varrho_k > 0 \\ J_1|_{E_k} = (e_2 e_1^* - e_1 e_2^*) & & J_2|_{E_k} = \pm J_1|_{E_k} & \\ \omega_1|_{E_k} = \lambda_k(e_1^* \wedge e_2^*) & & \omega_2|_{E_k} = \pm\varrho_k\omega_1|_{E_k} & \end{array} \tag{60}$$

where $e_2 = J_1 e_1$, e_1 is any given vector in E_k and e^* are the dual basis of e.[7]

Every linear vector field preserving both $h_1 = (g_1, \omega_1)$ and $h_2 = (g_2, \omega_2)$ will have a representative matrix commuting with those of T and G, and it will be block-diagonal in the same eigenspaces E_k. Therefore, in the generic case, the analysis can be restricted to each two-dimensional subspace E_k in which the vector field will preserve both a symplectic structure and a positive-definite metric. Therefore it will be in $sp(2) \cap SO(2) = U(1)$ and, on each E_k, it will represent a harmonic oscillator with frequencies depending in general on V_k.

Having discussed the general case, and to gather more insight into the problem we are discussing here, we will describe now in full details the two-dimensional case.

3.1. A two-dimensional example

Starting from the observation that two quadratic forms[8] can always be diagonalized simultaneously (at the price of using a non-orthogonal transformation, if necessary) we can assume from the start g_1 and g_2 to be of the form

$$g_1 = \begin{vmatrix} \varrho_1 & 0 \\ 0 & \varrho_2 \end{vmatrix} \tag{61}$$

[7] In other words, on each subspace g_1 and g_2 are proportional, while $J_1 = \pm J_2$ and accordingly $\omega_2 = \pm\varrho\omega_1$.

[8] One of which is assumed to be positive.

and

$$g_2 = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix}. \tag{62}$$

The more general J such that $J^2 = -1$ will be of the form

$$J = \begin{vmatrix} a & b \\ -\frac{(1+a^2)}{b} & -a \end{vmatrix}. \tag{63}$$

Compatibility with g_1 requires that J be anti-Hermitian (w.r.t. g_1), and this leads to

$$J = J_{1\pm} = \begin{vmatrix} 0 & \pm\sqrt{\frac{\varrho_2}{\varrho_1}} \\ \mp\sqrt{\frac{\varrho_1}{\varrho_2}} & 0 \end{vmatrix} \tag{64}$$

and similarly

$$J = J_{2\pm} = \begin{vmatrix} 0 & \pm\sqrt{\frac{\sigma_2}{\sigma_1}} \\ \mp\sqrt{\frac{\sigma_1}{\sigma_2}} & 0 \end{vmatrix} \tag{65}$$

from the requirement of admissibility with g_2.

As a consequence

$$\omega = \omega_{1\pm} = \begin{vmatrix} 0 & \pm\sqrt{\varrho_2\varrho_1} \\ \mp\sqrt{\varrho_2\varrho_1} & 0 \end{vmatrix} \tag{66}$$

and

$$\omega = \omega_{2\pm} = \begin{vmatrix} 0 & \pm\sqrt{\sigma_2\sigma_1} \\ \mp\sqrt{\sigma_2\sigma_1} & 0 \end{vmatrix}. \tag{67}$$

Now we have all the admissible structures, i.e. $(g_1, \omega_{1\pm}, J_{1\pm})$ and $(g_2, \omega_{2\pm}, J_{2\pm})$.

Let us compute the invariance group for the first triple having made a definite choice for the possible signs (say $J = J_+$). The group is easily seen to be

$$O_1(t) = \cos(t)\mathbb{I} + \sin(t)J_1 = \begin{vmatrix} \cos(t) & \sqrt{\frac{\varrho_2}{\varrho_1}}\sin(t) \\ -\sqrt{\frac{\varrho_1}{\varrho_2}}\sin(t) & \cos(t) \end{vmatrix} \tag{68}$$

while for the second triple we obtain

$$O_2(t) = \cos(t)\mathbb{I} + \sin(t)J_2 = \begin{vmatrix} \cos(t) & \sqrt{\frac{\sigma_2}{\sigma_1}}\sin(t) \\ -\sqrt{\frac{\sigma_1}{\sigma_2}}\sin(t) & \cos(t) \end{vmatrix} \tag{69}$$

and in general we obtain two different realizations of $SO(2)$.

The two realizations have only a trivial intersection (coinciding with the identity) if $\rho_2/\rho_1 \neq \sigma_2/\sigma_1$, and coincide when $\rho_2/\rho_1 = \sigma_2/\sigma_1$. The latter condition is easily seen (by imposing, e.g., $[J_{1,2}, T] = 0$) to be precisely the condition of compatibility of the two triples.

To conclude the discussion of the example, let us see what happens in the complexified version of the previous discussion.

To begin with, we have to define multiplication by complex numbers on $\mathbb{R}^2$, thus making it a complex vector space, and this can be done in two ways, namely as

$$(x + \mathrm{i}y)\begin{vmatrix} a \\ b \end{vmatrix} = (x\mathbb{I} + J_1 y)\begin{vmatrix} a \\ b \end{vmatrix} \tag{70}$$

or as

$$(x + \mathrm{i}y)\begin{vmatrix} a \\ b \end{vmatrix} = (x\mathbb{I} + J_2 y)\begin{vmatrix} a \\ b \end{vmatrix}. \tag{71}$$

Correspondingly, we can introduce two different Hermitian structures on $\mathbb{R}^2$ as

$$(\cdot, \cdot)_1 = g_1 + \mathrm{i}\omega_1 \tag{72}$$

or as

$$(\cdot, \cdot)_2 = g_2 + \mathrm{i}\omega_2. \tag{73}$$

They are antilinear in the first factor and in each case the corresponding multiplication by complex numbers must be used. The $O_1(t)$ and $O_2(t)$ actions both coincide with the multiplication of points of $\mathbb{R}^2$ by the complex numbers $\mathrm{e}^{\mathrm{i}t}$ (i.e. with different realizations of $U(1)$), but the definition of multiplication by complex numbers is different in the two cases.

Going back to the general case, we can make contact with the theory of complete integrability of a bi-Hamiltonian system by observing that T plays here the role of a recursion operator (see [2]). Indeed, we show now that it generates a basis of vector fields preserving both the Hermitian structures h_a given by

$$\Gamma_1, T\Gamma_1, \ldots, T^{n-1}\Gamma_1. \tag{74}$$

To begin with, these fields preserve all the geometrical structures, commute pairwise and are linearly independent. In fact these properties follow from the observation that T, being a constant 1–1 tensor, satisfies the Nijenhuis condition [6]. Therefore, for any vector field X

$$L_{TX}T = TL_X T \tag{75}$$

which, T being invertible, amounts to

$$L_{TX} = TL_X. \tag{76}$$

So, $\forall k \in \mathbb{N}$

$$L_{T^k\Gamma_1} = TL_{T^{k-1}\Gamma_1} = \cdots = T^k L_{\Gamma_1} \tag{77}$$

and

$$T^k L_{\Gamma_1}\omega_a = 0 = T^k L_{\Gamma_1} g_a. \tag{78}$$

Moreover, $\forall s \in \mathbb{N}$

$$\left[T^{k+s}\Gamma_1, T^k\Gamma_1\right] = L_{T^{k+s}\Gamma_1}T^k\Gamma_1 = T^s L_{T^k\Gamma_1}T^k\Gamma_1 = T^s\left[T^k\Gamma_1, T^k\Gamma_1\right] = 0. \tag{79}$$

Besides, the assumption of minimal degeneracy of T implies that the minimal polynomial [7] of T be of degree n. Indeed, we have shown that the diagonal form of T is

$$T = \bigoplus_{k=1,\ldots,n} \{\pm\rho_k\mathbb{I}_k\} \tag{80}$$

where $\mathbb{I}_k$ is the identity on V_k. Any linear combination

$$\sum_{r=0}^{m} \alpha_r T^r = 0 \qquad m \leqslant n-1 \tag{81}$$

yields a linear system for α_r of n equations in $m+1$ unknowns whose matrix of coefficients is of maximal rank and that, for $m = n-1$, coincides with the full Vandermonde matrix of ρ_k.

Then, we can conclude that the n vector fields $T^r\Gamma_1$, $r = 0, 1, \ldots, n-1$, form a basis.

4. The infinite-dimensional case

We now analyse the same kind of problems in the framework of quantum mechanics (QM), taking advantage of the experience and results we have obtained in the previous sections where we dealt with a real $2n$-dimensional vector space.

In QM, the Hilbert space $\mathbb{H}$ is given as a vector space over the field of complex numbers. Now we assume that two Hermitian structures are given on it, which we will denote as $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ (both linear, for instance, in the second factor). As in the real case, we look for the group that leaves both structures invariant, that is the group of unitary transformations w.r.t. both Hermitian structures. We call them 'bi-unitary'.

In order to assure that $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ do not define different topologies on $\mathbb{H}$, it is necessary that there exist $A, B \in \mathbb{R}$, $0 < A, B$, such that

$$A\,\|x\|_2 \leqslant \|x\|_1 \leqslant B\,\|x\|_2 \qquad \forall x \in \mathbb{H}. \tag{82}$$

The use of Riesz theorem on bounded linear functionals immediately implies that there exists an operator F defined implicitly by the equation

$$(x, y)_2 = (Fx, y)_1 \qquad \forall x, y \in \mathbb{H}. \tag{83}$$

F replaces the previous G and T tensors of the real vector space situation, i.e. now it contains both the real and imaginary parts of the Hermitian structure, and in fact

$$F = (g_1 + \mathrm{i}\omega_1)^{-1} \circ (g_2 + \mathrm{i}\omega_2). \tag{84}$$

It is trivial to show that F is bounded, positive and self-adjoint with respect to both Hermitian structures and that

$$\frac{1}{B^2} \leqslant \|F\|_1 \leqslant \frac{1}{A^2} \qquad \frac{1}{B^2} \leqslant \|F\|_2 \leqslant \frac{1}{A^2}. \tag{85}$$

If $\mathbb{H}$ is finite-dimensional, F can be diagonalized, the two Hermitian structures decompose in each eigenspace of F, where they are proportional and we get immediately that the group of bi-unitary transformations is indeed

$$U(n_1) \times U(n_2) \times \cdots \times U(n_k) \qquad n_1 + n_2 + \cdots + n_k = n = \dim \mathbb{H} \tag{86}$$

where n_i denotes the degeneracy of the ith eigenvalue of F.

In the infinite-dimensional case F may have a point part of the spectrum and a continuum part. From the point part of the spectrum, one gets $U(n_1) \times U(n_2) \times \cdots$ where now n_i can also be ∞. The continuum part is more delicate to discuss. It will contain for sure the commutative group U_F of bi-unitary operators of the form $\{\mathrm{e}^{\mathrm{i}f(F)}\}$, where f is any real-valued function (with very mild properties [8]).

The concept of genericity in the infinite-dimensional case cannot be given as easily as in the finite-dimensional case. One can say that the eigenvalues should be nondegenerate but what for the continuous spectrum? We give here an alternative definition that works for the finite and infinite cases as well.

Note first that any bi-unitary operator must commute with F. Indeed $(x, U^\dagger F U y)_2 = (Ux, FUy)_2 = (FUx, Uy)_2 = (Ux, Uy)_1 = (x, y)_1 = (Fx, y)_2 = (x, Fy)_2$, from this $U^\dagger F U = F$, $[F, U] = 0$.

The group of bi-unitary operators therefore belongs to the commutant F' of the operator F. The genericity condition can be restated in a purely algebraic form as follows:

Definition. *Two Hermitian forms are in a generic position iff $F'' = F'$, i.e. the bicommutant of F coincides with the commutant of F.*

In other words, this means that F generates a complete set of observables.

This definition reduces, for the case of a pure point spectrum, to the condition of nondegeneracy of the eigenvalues of F and, in the real case, to the minimum possible degeneracy of the eigenvalues of T and G, that is 2.

To grasp how the definition works, we will give some simple examples. Consider $(F\psi)(x) = x^2\psi(x)$ on the space $L_2([-b, -a] \cup [a, b])$ with $0 < a < b$; then the operator x, its powers x^n and the parity operator P belong to F' while F'' does not contain x (and any odd power of x) because they do not commute with P. So if $F = x^2$ the two Hermitian structures are not in a generic position because $F'' \subset F'$. In contrast, on the space $L_2([a, b])$, $F'' = F'$ because a parity operator P does not exist in this case, so the two Hermitian structures are now in a generic position. In this case, the group of bi-unitary operators is $\left\{e^{if(x^2)t}\right\}$ for the appropriate class of functions f. In some sense, when a continuous part of the spectrum is considered, there appears a continuous family of $U(1)$ as a counterpart of the discrete family of $U(1)$ corresponding to the discrete part of the spectrum.

Remarks.

(i) Suppose that complex Hilbert spaces with two Hermitian structures have been constructed from a given real vector space V using two compatible and admissible triples (g_1, ω_1, J_1) and (g_2, ω_2, J_2). Then, by complexification, we get two different Hilbert spaces, each one with its proper multiplication by complex numbers and with its proper Hermitian structure. The previous case we have just studied is obtained if we assume $J_1 = J_2$. It is easy to show that this is a sufficient condition for compatibility. This is the reason why in the quantum-mechanical case the group of bi-unitary transformations is never empty, and the compatibility condition is encoded already in the assumptions.

(ii) If $J_1 \neq J_2$ but the compatibility condition still holds, we know that V splits into $V_+ \oplus V_-$, where $J_1 = \pm J_2$ on $V_\pm$ respectively. On V_+ we have the previous case, while on V_- we get two Hermitian structures, one $\mathbb{C}$-linear and one anti-$\mathbb{C}$-linear in the second factor (which one is linear and which is antilinear depends on the complexification we have decided to use). From the point of view of the group of unitary transformations, this circumstance is irrelevant, because the set of unitary transformations does not change from being defined w.r.t. a Hermitian structure or w.r.t. its complex conjugate. We conclude from this that our analysis goes through in general, provided the compatibility condition holds.

5. Conclusions

We will try now to summarize our main result, by restating it at the same time in a more concise group-theoretical language. What we have shown is, to begin with, that once two admissible triples (g_1, ω_1, J_1) and (g_2, ω_2, J_2) are given on a real, even-dimensional vector space $V \approx \mathbb{R}^{2n}$, they define two $2n$-dimensional real representations $U_r(2n; g_1, \omega_1)$ and $U_r(2n; g_2, \omega_2)$ of the unitary group $U(n)$, $U_r(2n; g_a, \omega_a)$ $(a = 1, 2)$ being the group of transformations that simultaneously leave g_a and ω_a (and hence J_a) invariant. Their intersection

$$W_r =: \{U_r(2n; g_1, \omega_1) \cap U_r(2n; g_2, \omega_2)\} \tag{87}$$

will be their common subgroup that is an invariance group for both triples. The assumption of compatibility[9] implies that W_r should not reduce to the identity alone.

If the two triples are in a generic position, then

$$W_r = \underbrace{SO(2) \times SO(2) \times \cdots \times SO(2)}_{n \text{ factors}} \tag{88}$$

[9] As the previous two-dimensional example shows explicitly, but it should also be clear by now in general.

where $SO(2) \approx U(1)$ or more generally if the genericity assumption is dropped

$$W_r = U_r(2r_1; g, \omega) \times U_r(2r_2; g, \omega) \times \cdots \times U_r(2r_k; g, \omega) \tag{89}$$

where $r_1 + r_2 + \cdots + r_k = n$ and (g, ω) is any one of the two pairs (g_1, ω_1) and (g_2, ω_2).

The real vector space $V \approx \mathbb{R}^{2n}$ will then decompose into a direct sum of even-dimensional subspaces that are mutually orthogonal w.r.t. both metrics, and on each subspace the corresponding (realization of the) special orthogonal group will act irreducibly.

Alternatively, we can complexify $V \approx \mathbb{R}^{2n}$, and that in two different ways, using the two complex structures that are at our disposal. The equivalent statement in the complex framework will then be as follows.

Given two Hermitian structures h_a, $a = 1, 2$, on a complex n-dimensional vector space $\mathbb{C}^n$, they define two representations $U(n; h_a)$, $a = 1, 2$, of the group $U(n)$ on the same $\mathbb{C}^n$. $U(h_1, n)$ (respectively $U(h_2, n)$) will be the group of transformations that are unitary with respect to h_1 (respectively h_2). The group W of simultaneous invariance for both Hermitian structures,

$$W \equiv \{U(h_1, n) \cap U(h_2, n)\} \tag{90}$$

will be a subgroup of both $U(h_1, n)$ and $U(h_2, n)$, and our assumption of compatibility of h_a implies that the component of W connected to the identity should not reduce to the identity alone.

The assumption of genericity implies that

$$W = \underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{n \text{ factors}} \tag{91}$$

If the assumption of genericity is dropped, one can easily show, along the same lines as in the generic case, that W will be of the form

$$W = U(r_1) \times U(r_2) \times \cdots \times U(r_k) \tag{92}$$

with $r_1 + r_2 + \cdots + r_k = n$. $\mathbb{C}^n$ will decompose accordingly into a direct sum of subspaces that will be mutually orthogonal with respect to both h_a, and on each subspace the appropriate $U(r)$ will act irreducibly.

We have also shown that these results generalize to the infinite-dimensional case as well. Some extra assumptions must be added on the Hermitian structures in order that they define the same topology in $\mathbb{H}$ and an appropriate definition of genericity must also be given. Then, a decomposition as in equations (91) and (92) is obtained, possibly with denumerable discrete terms and a continuum part as well. We note that, in the spirit of this work where two Hermitian structures are given from the very beginning, it is natural to supplement the compatibility condition, in the infinite-dimensional case, with a topological equivalence condition. However, from the point of view of the study of bi-Hamiltonian systems, where a fixed dynamics is given, it would be more natural to assume some weaker regularity condition, for instance, that the given dynamics should be continuous with respect to both structures.

Recently, bi-Hamiltonian systems ‘generated’ out of a pencil of compatible Poisson structures have been considered [9], also in connection with the separability problem [10]. It should be noted that our compatible structures would give rise to a pencil of compatible triples defined by

$$g_\gamma = g_1 + \gamma g_2 \qquad \omega_\gamma = \omega_1 + \gamma \omega_2 \qquad J_\gamma. \tag{93}$$

A systematic comparison with this approach is presently under consideration.

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